Complex Number (Problem Set)

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1. Suppose $A(z_1), B(z_2), C(z_3), D(z_4)$ are 4 points in the argand plane such that AB is perpendicular to CD. Prove that $\frac{z_1-z_2}{z_3-z_4}$ is purely imaginary.

2. Show that the point a' is the reflection of the point a in the line $z\bar{b}+\bar{z}b+c=0$, if $a'\bar{b}+\bar{z}a'+c=0$.

3. If $|z-2+i| \leq 2$, find the greatest and least value of $|z|$. (Can you do it without algebra?)

4. If ω is complex cube root of unity and a, b, c are such that
$$\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}=2\omega^2$$
and
$$\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}=2\omega$$
then find the value of
$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}.$$

5. If a, b, c are distinct integers and $\omega(\neq 1)$ is a cube root of unity, find minimum value of
$$|a+b\omega+c\omega^2|+|a+b\omega^2+c\omega|$$

6. If A, B and C represent the complex numbers z_1, z_2 and z_3 respectively on the complex plane and the angles at B and C are each equal to $\frac{1}{2}(\pi-\alpha)$, then prove that
$$(z_2-z_3)^2=4(z_3-z_1)(z_1-z_2)\sin^2\left(\frac{\alpha}{2}\right).$$

7. If z_1 and z_2 are two complex numbers such that
$$\left|\frac{z_1-z_2}{z_1+z_2}\right|=1,$$
then prove that
$$\frac{iz_1}{z_2}=k, \text{ where } k \text{ is a real number.}$$
Find the angle between the lines from the origin to the points z_1+z_2 and z_1-z_2 in terms of k.

8. If a is a complex number such that $|a|=1$, then find the value of a, so that the equation $az^2+z+1=0$ has one purely imaginary root.

9. If $n \in \mathbb{N} > 1$, find the sum of real parts of the roots of the equation $z^n=(z+1)^n$.

10. Interpret the following equations geometrically on the Argand plane.
    (i) $|z-1|+|z+1|=4$
    (ii) $\arg(z+i)-\arg(z-i)=\dfrac{\pi}{2}$
    (iii) $1<|z-2-3i|<4$

(iv) $\dfrac{\pi}{4} < \arg(z) < \dfrac{\pi}{3}$

(v) $\log_{\cos\frac{\pi}{3}} \left\{ \dfrac{|z-1|+4}{3|z-1|-2} \right\} > 1$

11. If ω is the nth root of unity and z_1, z_2 are any two complex numbers, then prove that

$$\sum_{k=0}^{n-1} \left|z_1 + \omega^k z_2\right|^2 = n\left\{|z_1|^2 + |z_2|^2\right\}, \quad \text{where } n \in \mathbb{N}.$$

12. Let $\sum_{i=1}^{4} a_i = 0$ and $\sum_{j=1}^{4} a_j z_j = 0$, then prove that z_1, z_2, z_3 and z_4 are concyclic, if

$$a_1 a_2 |z_1 - z_2|^2 = a_3 a_4 |z_3 - z_4|^2.$$

13. Prove that all roots of $11z^{10} + 10iz^9 + 10iz - 11 = 0$ have unit modulus (or equivalent $|z| = 1$).

14. Let a,b,c be real numbers. Let z_1,z_2,z_3 be complex numbers such that $|z_k| = 1$ $(k = 1, 2, 3)$ and $\frac{z_1}{z_2} + \frac{z_2}{z_3} + \frac{z_3}{z_1} = 1$ Find $|az_1 + bz_2 + cz_3|$.

15. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lie on the circles $(x-x_0)^2+(y-y_0)^2 = r^2$ and $(x-x_0)^2+(y-y_0)^2 = 4r^2$ respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| = ?$

16. If $|z| \geq 3$, then determine the least value of $|z + \frac{1}{z}|$.

17. If $|z + \frac{4}{z}| = 2$, then find the minimum and maximum values of $|z|$.

18. For any integer k, let $\alpha_k = \cos(\frac{k\pi}{7}) + i\sin(\frac{k\pi}{7})$. Then find the value of $\frac{\sum_{i=1}^{12} |\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3} |\alpha_{4k-1} - \alpha_{4k-2}|}$.

19. Let $a, b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Suppose $S = \{z \in \mathbb{C} : z = \frac{1}{a+ibt}, t \in \mathbb{R}, t \neq 0\}$. If $z = x + iy$ and $z \in S$, find locus of z.

20. Let z be a complex number satisfying $|z|^3 + 2z^2 + 4\bar{z} - 8 = 0$, where$\bar{z}$ denotes the complex conjugate of z . Let the imaginary part of z be nonzero. $|z|^2$=? $|z - \bar{z}|^2 = ?|z|^2 + |z + \bar{z}|^2 = ?$ $|z + 1|^2 = ?$

21. $z_1, z_2, \cdots$ be a sequence of complex numbers defined by $z_1 = i$ and $z_{n+1} = z_n^2 + i$ for $n \geq 1$. Then $|z_{2024} - z_1|$ =?

22. Consider the two subsets of $\mathbb{C}$, $A = \{\frac{1}{z} : |z| = 2\}$ and $B = \{\frac{1}{z} : |z - 1| = 2\}$. Identify A and B as geometric figures.

23. Find the minimum possible value of $|z|^2 + |z - 3|^2 + |z - 6i|^2$, where z is a complex number.

24. Let $0 < a_1 < a_2 < \cdots < a_n$ be real numbers. Show that the equation $\frac{a_1}{a_1 - x} + \frac{a_2}{a_2 - x} + \cdots + \frac{a_n}{a_n - x} = 2015$ has exactly n real roots.

25. If z_1, z_2, z_3 are non-zero complex numbers such that

$$\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}.$$

Then prove that z_1, z_2, z_3 lie on a circle passing through the origin.

26. What is the area of the figure in the complex plane enclosed by the origin and the set of all points $\frac{1}{z}$ such that

$$(1 - 2i)z + (-2i - 1)\overline{z} = 6i?$$

27. What is the area of the region in the complex plane consisting of all points z satisfying both

$$\left|\frac{1}{z} - 1\right| < 1 \quad \text{and} \quad |z - 1| < 1?$$

($|z|$ denotes the magnitude of a complex number, i.e. $|a + bi| = \sqrt{a^2 + b^2}$.)

28. Let $z = x + iy$ be a complex number with x and y rational and with $|z| = 1$.

(a) Find two such complex numbers. (b) Show that $|z^{2n} - 1| = 2|\sin n\theta|$, where $z = e^{i\theta}$.